

Spinor helicity structures in higher spin theories

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Abstract

It is shown that the coefficient of the cubic interaction vertex, in higher spin Lagrangians, has a very simple form when written in terms of spinor helicity products. The result for a higher-spin field, of spin λ , is equal to the corresponding Yang-Mills coefficient raised to the power λ . Among other things, this suggests perturbative ties, similar to the KLT relations, between higher spin theories and pure Yang-Mills. This result is obtained in four-dimensional flat spacetime.

Introduction

In four-dimensional flat spacetime, Lagrangians describing fields of arbitrary spin have been investigated in detail [1, 2]. In particular, there exist consistent Lagrangians, to cubic order, for higher spin fields ($\lambda > 2$). In this letter, we observe that the coefficient of the cubic interaction vertex in higher spin Lagrangians is equal to the corresponding coefficient in pure Yang-Mills theory raised to the power λ , ie.

$$L_3^\lambda = \left[\frac{\langle k l \rangle^3}{\langle l p \rangle \langle p k \rangle} \right]^\lambda, \quad (1)$$

in terms of notation described below. This is an off-shell result, valid at the level of the Lagrangian, and interesting because it suggests that the spinor helicity approach [3] that has been so successful in studies of spin 1 and spin 2 theories [4] may prove useful in understanding higher spin theories [5] as well. For interesting work on related issues see [6].

The result also suggests that there exist KLT-like relations [7] that extend beyond the Yang-Mills – Gravity system [8, 9].

Notation

When studying theories based on helicity considerations, it is natural to work in light-cone gauge where only helicity states propagate. With the metric $(-, +, +, +)$, we define

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_3). \quad (2)$$

x^+ plays the role of light-cone time and ∂_+ of the light-cone Hamiltonian. ∂_- is now a spatial derivative and its inverse, $\frac{1}{\partial_-}$, is defined using the prescription in [10]. We also define

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(x^1 + i x^2), \quad \bar{\partial} \equiv \frac{\partial}{\partial x} = \frac{1}{\sqrt{2}}(\partial_1 - i \partial_2), \\ \bar{x} &= \frac{1}{\sqrt{2}}(x^1 - i x^2), \quad \partial \equiv \frac{\partial}{\partial \bar{x}} = \frac{1}{\sqrt{2}}(\partial_1 + i \partial_2). \end{aligned} \quad (3)$$

A four-vector p_μ may be expressed as a bispinor $p_{a\dot{a}}$ using the $\sigma^\mu = (-\mathbf{1}, \sigma)$ matrices

$$p_{a\dot{a}} \equiv p_\mu (\sigma^\mu)_{a\dot{a}} = \begin{pmatrix} -p_0 + p_3 & p_1 - i p_2 \\ p_1 + i p_2 & -p_0 - p_3 \end{pmatrix} = \sqrt{2} \begin{pmatrix} -p_- & \bar{p} \\ p & -p_+ \end{pmatrix}. \quad (4)$$

The determinant of this matrix is

$$\det(p_{a\dot{a}}) = -2(p\bar{p} - p_+p_-) = -p^\mu p_\mu. \quad (5)$$

For light-like p_μ , $p_+ = \frac{p\bar{p}}{p_-}$ represents the on-shell condition, $p^2 = 0$. We then define holomorphic and anti-holomorphic spinors

$$\lambda_a = \frac{2^{\frac{1}{4}}}{\sqrt{p_-}} \begin{pmatrix} p_- \\ -p \end{pmatrix}, \quad \tilde{\lambda}_{\dot{a}} = -(\lambda_a)^* = -\frac{2^{\frac{1}{4}}}{\sqrt{p_-}} \begin{pmatrix} p_- \\ -\bar{p} \end{pmatrix}, \quad (6)$$

such that $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ on-shell. The off-shell holomorphic and anti-holomorphic spinor products read

$$\langle i j \rangle = \sqrt{2} \frac{p_-^i p_-^j - p_-^j p_-^i}{\sqrt{p_-^i p_-^j}}, \quad [i j] = \sqrt{2} \frac{\bar{p}_-^i \bar{p}_-^j - \bar{p}_-^j \bar{p}_-^i}{\sqrt{p_-^i p_-^j}}. \quad (7)$$

Higher Spin Lagrangians

The light-cone action for a field ϕ (and its conjugate $\bar{\phi}$) of spin λ , to cubic order, reads [1]

$$S = \int d^4x \left\{ \frac{1}{2} \bar{\phi}^a \square \phi^a + \alpha f^{abc} \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \left[\bar{\phi}^a \partial_-^\lambda \left(\frac{\bar{\partial}^{(\lambda-n)}}{\partial_-^{(\lambda-n)}} \phi^b \frac{\bar{\partial}^n}{\partial_-^n} \phi^c \right) + c.c \right] + O(\alpha^2) \right\}, \quad (8)$$

for λ odd and

$$S = \int d^4x \left\{ \frac{1}{2} \bar{\phi} \square \phi + \alpha \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \left[\bar{\phi} \partial_-^\lambda \left(\frac{\bar{\partial}^{(\lambda-n)}}{\partial_-^{(\lambda-n)}} \phi \frac{\bar{\partial}^n}{\partial_-^n} \phi \right) + c.c \right] + O(\alpha^2) \right\}, \quad (9)$$

for λ even. Note that interactions involving fields of odd λ are accompanied by antisymmetric structure constants. Since ϕ and $\bar{\phi}$ have definite helicities, both Lagrangians have the following helicity structure

$$L \sim L_{+-} + \alpha L_{-++} + \alpha L_{+--} + O(\alpha^2). \quad (10)$$

The first cubic vertex in (10) is non-MHV in structure and could, in principle, be eliminated by a suitable field redefinition if our aim were to produce an MHV Lagrangian [8]. However, since the focus of this letter is on the structure of cubic interaction vertices, this is not necessary. To obtain (1), it is sufficient to focus on the L_{+--} vertex in momentum space. The $\lambda = 1$ and $\lambda = 2$ versions of (1) are well known so we concentrate on spin 3 and spin 4 fields.

Spin 3

From (8), the cubic interaction vertex for a spin 3 field in momentum space reads

$$L_{+--}^{\lambda=3} = \int d^4p d^4k d^4l (k_- + l_-)^3 \left\{ \frac{k_-^3}{2k_-^3} - \frac{l_-^3}{2l_-^3} + \frac{3kl^2}{2k_- l_-^2} - \frac{3k^2 l}{2k_-^2 l_-} \right\} \\ \times \delta^4(p+k+l) f^{abc} \phi^a(p) \bar{\phi}^b(k) \bar{\phi}^c(l). \quad (11)$$

This formula has been made manifestly antisymmetric in the momenta k and l . Recasting (11) in terms of the off-shell holomorphic spinor product in (7) is straightforward and yields

$$L_{+--}^{\lambda=3} = \int d^4p d^4k d^4l \sqrt{2} \frac{\langle k l \rangle^9}{\langle l p \rangle^3 \langle p k \rangle^3} \delta^4(p+k+l) f^{abc} \phi^a(p) \bar{\phi}^b(k) \bar{\phi}^c(l). \quad (12)$$

Spin 4

The cubic interaction vertex for a spin 4 field, in momentum space, from (9) reads

$$L_{+--}^{\lambda=4} = \int d^4p d^4k d^4l (k_- + l_-)^4 \left\{ \frac{k_-^4}{2k_-^4} + \frac{l_-^4}{2l_-^4} - \frac{4kl^3}{k_-l_-^3} - \frac{4k^3l}{k_-^3l_-} + \frac{6k^2l^2}{k_-^2l_-^2} \right\} \\ \times \delta^4(p+k+l) \phi(p) \bar{\phi}(k) \bar{\phi}(l) , \quad (13)$$

this expression having been made symmetric in k and l . This simplifies to

$$L_{+--}^{\lambda=4} = \int d^4p d^4k d^4l \frac{1}{4} \frac{\langle kl \rangle^{12}}{\langle lp \rangle^4 \langle pk \rangle^4} \delta^4(p+k+l) \phi(p) \bar{\phi}(k) \bar{\phi}(l) . \quad (14)$$

The derivative structures in (8) and (9) strongly suggest that the relationship (1) holds for all spins, as far as the cubic vertex is concerned.

* * *

An interesting question is whether relations of the type in (1) can be established for higher order vertices in the Lagrangian. If so, do they define a consistent interacting tree-level S-matrix with the usual properties? ¹ For related discussions see [11]. Also, do spinor helicity structures simply put in an appearance at order α or are they an indication that higher spin theories possess interesting mathematical structures, similar to those that arise in Yang-Mills theory such as MHV amplitudes. Finally, just as gravity is in some ways the “square” of Yang-Mills [12]

$$\text{Gravity} \sim (\text{Yang-Mills}) \times (\text{Yang-Mills}) ,$$

formula (1) could be an indication that

$$(\text{Spin}=\lambda) \sim (\text{Yang-Mills})^\lambda .$$

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